Principles of Mathematics 12

January 1999 Provincial Examination

ANSWER KEY / SCORING GUIDE

CURRICULUM:

Organizers	Sub-Organizers
1. Problem Solving	A Problem Set
2. Patterns and Relations	B Sequences and Series
	C Polynomials
	D Logarithms and Exponents
	E Quadratic Relations
	F Quadratic Systems
3. Shape and Space	G Trigonometry
	H Geometry

Part A: Multiple Choice

Q	K	C	CO	PLO	Q	K	C	CO	PLO
1.	B	K	2	C4	24.	A	K	2	B1
2.	C	U	2	C5	25.	D	U	2	B4
3.	D	U	2	C4	26.	C	U	2	B2
4.	A	U	2, 1	C6, A7	27.	C	U	2	B4
5.	D	H	2	C1	28.	A	H	2	B4, 6
6.	B	K	2	E5	29.	C	K	3	G1
7.	A	K	2	E2	30.	D	U	3	G2
8.	D	U	2	E5	31.	C	U	3	G3
9.	B	K	2	F1	32.	A	U	3	G6, A7
10.	A	U	2	F4	33.	A	U	3	G5
11.	D	U	2	F3	34.	C	U	3	G5
12.	B	U	2	E4	35.	D	U	3	G8
13.	B	U	2	E6	36.	B	U	3	G9
14.	B	H	2	F1	37.	D	H	3	G2
15.	B	H	2, 1	F1, A7	38.	A	H	3	G3, 7
16.	D	H	2	E7	39.	B	U	3	H1
17.	C	U	2	D5	40.	B	U	3	H1
18.	A	U	2	D5	41.	C	U	3	H3
19.	B	K	2	D1	42.	C	H	3	H1
20.	A	U	2	D2	43.	D	U	1	A3
21.	A	U	2	D5	44.	C	U	1	A4
22.	B	H	2	D5	45.	D	H	1	A1
23.	C	H	2	D5					

Multiple Choice = 45 marks

Part B: Written Response

Q	B	C	S	CO	PLO
1.	1	U	3	2	C9, A7
2.	2	U	3	2	D6
3.	3	U	3	2	E4
4.	4	U	3	2	B7
5.	5	U	3	3	G8
6.	6	U	3	1	A3, 7
7.	7	U	3	3	H2
8.	8	H	4	3	H2

Written Response = 25 marks

Multiple Choice = 45 (45 questions)
Written Response = 25 (8 questions)
EXAMINATION TOTAL = 70 marks

LEGEND:

Q = Question Number
K = Keyed Response
C = Cognitive Level
B = Score Box Number
S = Score
CO = Curriculum Organizer
PLO = Prescribed Learning Outcome

PART B: WRITTEN RESPONSE

Value: 25 marks **Suggested Time: 45 minutes**

INSTRUCTIONS: Rough-work space has been incorporated into the space allowed for answering each question. You may not need all the space provided to answer each question. Where required, place the final answer for each question in the space provided.

If, in a justification, you refer to information produced by the calculator, this information must be presented clearly in the response. For example, if a graph is used in the solution of the problem, it is important to sketch the graph, showing its general shape and indicating the appropriate window dimensions.

When using the calculator, you should provide a decimal answer that is correct to **at least two decimal places** (unless otherwise indicated). Such rounding should occur **only** in the final step of the solution.

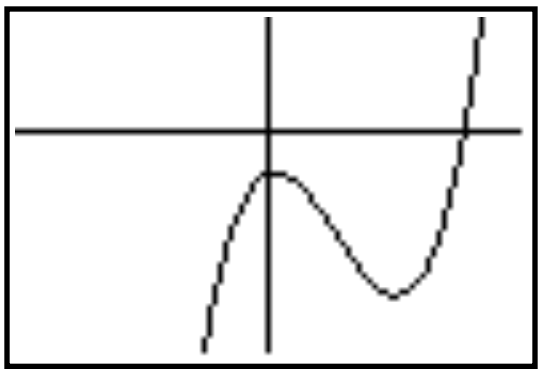
Full marks will NOT be given for the final answer only.

1. Solve the following inequality using a graphing calculator. **(3 marks)**

$$x^3 - 8x^2 \geq -4x + 20$$

Sketch the graph in the viewing window below and indicate appropriate window dimensions. State the function(s) used in your graph. Ensure that the relative maximum and relative minimum points of the function(s) are visible within the viewing window. The solution may be given in algebraic form or shown on a number line.

Solution



$Y_1 = x^3 - 8x^2 + 4x - 20$ ← $\frac{1}{2}$ **mark** for equation

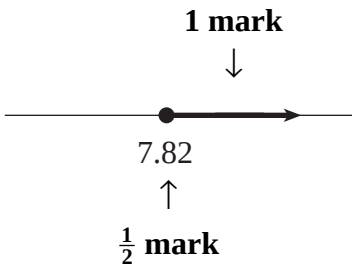
← $\frac{1}{2}$ **mark** for graph

$x\,[-10,\ 10]$ $y\,[-100,\ 50]$ ← $\frac{1}{2}$ **mark** for window dimensions

$x \geq 7.82$ ← $\frac{1}{2}$ **mark** for 7.82
1 mark for $\geq$

OR

1 mark



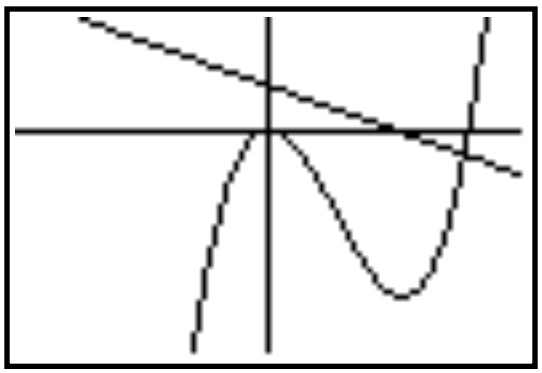
$\frac{1}{2}$ **mark**

1. Solve the following inequality using a graphing calculator. **(3 marks)**

$$x^3 - 8x^2 \geq -4x + 20$$

Sketch the graph in the viewing window below and indicate appropriate window dimensions. State the function(s) used in your graph. Ensure that the relative maximum and relative minimum points of the function(s) are visible within the viewing window. The solution may be given in algebraic form or shown on a number line.

Alternate Solution



$Y_1 = x^3 - 8x^2$
$Y_2 = -4x + 20$
} ← $\frac{1}{2}$ **mark** for equations

← $\frac{1}{2}$ **mark** for graph

$x\,[-10,\ 10]$ $y\,[-100,\ 50]$ ← $\frac{1}{2}$ **mark** for window dimensions

$x \geq 7.82$ } ← $\frac{1}{2}$ **mark** for 7.82
1 mark for $\geq$

OR

1 mark
↓

7.82
↑
$\frac{1}{2}$ **mark**

2. A population of frogs doubles every 20 weeks. If the present population is 400 frogs, how long will it take for the population to reach 10 000 ? **(3 marks)**

Solution

Method 1:

n = number of weeks required

$10\,000 = 400(2)^{\frac{n}{20}}$ ← **1 $\frac{1}{2}$ marks**

$25 = 2^{\frac{n}{20}}$

$\log(25) = \log(2)^{\frac{n}{20}}$

$\log(25) = \frac{n}{20}\log(2)$

$n = \frac{20\log 25}{\log 2}$

} ← **1 mark** for any appropriate logarithm statement

$n = 92.88$ weeks ← **$\frac{1}{2}$ mark**

Alternate Solution

Method 1:

n = number of weeks required

$10\,000 = 400(2)^{\frac{n}{20}}$ ← **1 $\frac{1}{2}$ marks**

$25 = 2^{\frac{n}{20}}$

$\log_2 25 = \frac{n}{20}$ ← **1 mark**

$20\log_2 25 = n$

$92.88 \text{ weeks} = n$ ← **$\frac{1}{2}$ mark**

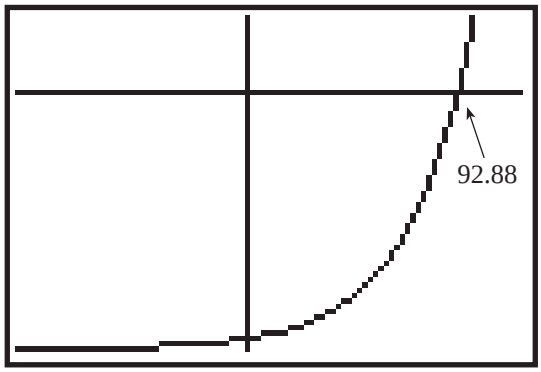
2. A population of frogs doubles every 20 weeks. If the present population is 400 frogs, how long will it take for the population to reach 10 000 ? **(3 marks)**

Solution

Method 2:

The equation $10\,000 = 400(2)^{\frac{n}{20}}$ **(1 $\frac{1}{2}$ marks)** could be solved using a graphing calculator.

e.g. $y = 400(2)^{\frac{x}{20}} - 10\,000$ and find the x-intercept.



$Y_1 = 400(2)^{\frac{x}{20}} - 10\,000$

$x = 92.88$ $\leftarrow$ **1 $\frac{1}{2}$ marks**

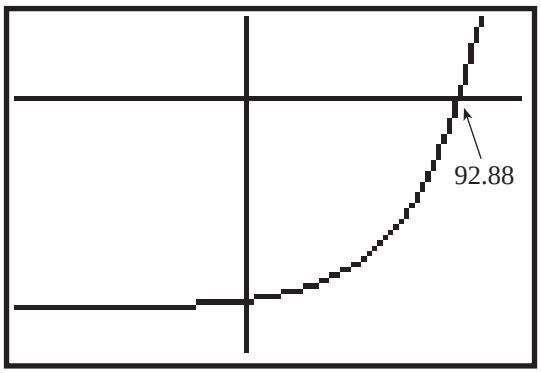
x $[-100,\ 120]$ y $[-10\,000,\ 2\,800]$

Alternate Solution

Method 2:

The equation $25 = (2)^{\frac{n}{20}}$ **(1 $\frac{1}{2}$ marks)** could be solved using a graphing calculator.

e.g. $y = (2)^{\frac{x}{20}} - 25$ and find the x-intercept.



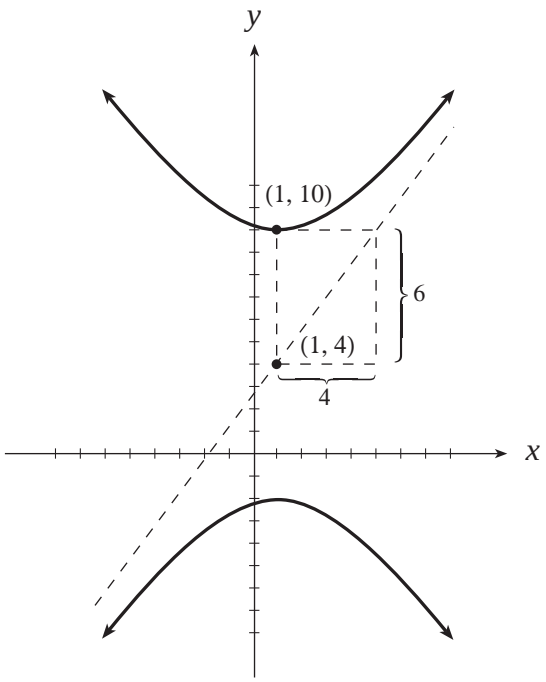
$Y_1 = (2)^{\frac{x}{20}} - 25$

$x = 92.88$ $\leftarrow$ **1 $\frac{1}{2}$ marks**

x $[-100,\ 120]$ y $[-30,\ 10]$

3. A hyperbola has centre $(1,\ 4)$, and one vertex at $(1,\ 10)$. If the asymptotes have slopes $\pm\frac{3}{2}$, determine the equation of the hyperbola in standard form. **(3 marks)**

Solution



$\frac{1}{2}$ **mark** ↓

$$\frac{(x-1)^2}{16} - \frac{(y-4)^2}{36} = -1 \quad \leftarrow \textbf{1 mark}$$

↑ **1 mark** ↑ $\frac{1}{2}$ **mark**

$\left\{\begin{array}{l} \frac{1}{2} \textbf{ mark} \text{ standard form} \\ \frac{1}{2} \textbf{ mark} \text{ correct hyperbola} \end{array}\right.$

OR

$\frac{1}{2}$ **mark** ↓

$$\frac{(y-4)^2}{36} - \frac{(x-1)^2}{16} = 1 \quad \leftarrow \textbf{1 mark}$$

↑ $\frac{1}{2}$ **mark** ↑ **1 mark**

$\left\{\begin{array}{l} \frac{1}{2} \textbf{ mark} \text{ standard form} \\ \frac{1}{2} \textbf{ mark} \text{ correct hyperbola} \end{array}\right.$

4. Pat's new job paid $\$x$ for the first month, and increased by \$50 each month after the first month. In a total of 24 months, including the first month, Pat earned \$60 000. Determine the value of x. **(3 marks)**

Solution

$x + x + 50 + x + 2(50) + ... + x + 23(50) = 60\ 000$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$a = x$

$d = 50$

$n = 24$

$S_{24} = 60\ 000$

$\leftarrow$ **1 $\frac{1}{2}$ marks**

$\frac{24}{2}(2x + (24-1)50) = 60\ 000$ $\leftarrow$ **1 mark**

$12(2x + 1\,150) = 60\ 000$

$2x + 1\,150 = 5\ 000$

$2x = 3\,850$

$x = 1\,925$ $\leftarrow$ **$\frac{1}{2}$ mark**

4. Pat's new job paid $\$x$ for the first month, and increased by \$50 each month after the first month. In a total of 24 months, including the first month, Pat earned \$60 000. Determine the value of x. **(3 marks)**

Alternate Solution

$x, \quad x+50, \quad x+100, \quad \ldots \quad x+1150$

1 2 3 24

$$S_n = \frac{n}{2}(a+\ell) \qquad a = x, \quad n = 24, \quad S_{24} = 60\,000, \quad \ell = x+1150$$

← **1 $\frac{1}{2}$ marks**

$$60\,000 = \frac{24}{2}(x + x + 1150) \qquad \leftarrow \textbf{1 mark}$$

$$5\,000 = 2x + 1150$$

$$2x = 3\,850$$

$$x = 1\,925 \qquad \leftarrow \textbf{$\frac{1}{2}$ mark}$$

5. Prove the following identity: **(3 marks)**

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

Solution

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

LEFT SIDE	RIGHT SIDE
	$= \dfrac{\left(\frac{1}{\sin\theta} + 1\right)\sin\theta}{\left(\frac{\cos\theta}{\sin\theta}\right)\sin\theta}$ ← **½ mark**
½ mark → $= \dfrac{\left(\frac{\cos\theta}{\sin\theta}\right)\sin\theta}{\left(\frac{1}{\sin\theta} - 1\right)\sin\theta}$	
½ mark → $= \dfrac{\cos\theta}{1 - \sin\theta}$	$= \dfrac{1 + \sin\theta}{\cos\theta} \dfrac{(1 - \sin\theta)}{(1 - \sin\theta)}$ ← **½ mark**
	$= \dfrac{1 - \sin^2\theta}{\cos\theta(1 - \sin\theta)}$
	$= \dfrac{\cos^2\theta}{\cos\theta(1 - \sin\theta)}$ ← **½ mark**
	$= \dfrac{\cos\theta}{1 - \sin\theta}$ ← **½ mark**

LS = RS

5. Prove the following identity: **(3 marks)**

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

Alternate Solution 1

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

LEFT SIDE	RIGHT SIDE
1 mark $\rightarrow \quad = \dfrac{(\csc\theta + 1)}{(\csc\theta + 1)} \dfrac{\cot\theta}{(\csc\theta - 1)}$	
$\frac{1}{2}$ **mark** $\rightarrow \quad = \dfrac{\cot\theta(\csc\theta + 1)}{\csc^2\theta - 1}$	
1 mark $\rightarrow \quad = \dfrac{\cot\theta(\csc\theta + 1)}{\cot^2\theta}$	
$\frac{1}{2}$ **mark** $\rightarrow \quad = \dfrac{\csc\theta + 1}{\cot\theta}$	

LS = RS

5. Prove the following identity: **(3 marks)**

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

Alternate Solution 2

$$\frac{\cot\theta}{\csc\theta - 1} = \frac{\csc\theta + 1}{\cot\theta}$$

LEFT SIDE	RIGHT SIDE
$\frac{1}{2}$ **mark** $\rightarrow \quad = \frac{(\csc\theta + 1)}{\cot\theta} \cdot \frac{\cot\theta}{\cot\theta}$	
1 mark $\rightarrow \quad = \frac{(\csc\theta + 1) \cdot \cot\theta}{\cot^2\theta}$	
$\frac{1}{2}$ **mark** $\rightarrow \quad = \frac{(\csc\theta + 1) \cdot \cot\theta}{\csc^2\theta - 1}$	
$\frac{1}{2}$ **mark** $\rightarrow \quad = \frac{(\csc\theta + 1)\cot\theta}{(\csc\theta + 1)(\csc\theta - 1)}$	
$\frac{1}{2}$ **mark** $\rightarrow \quad \frac{\cot\theta}{\csc\theta - 1}$	

LS = RS

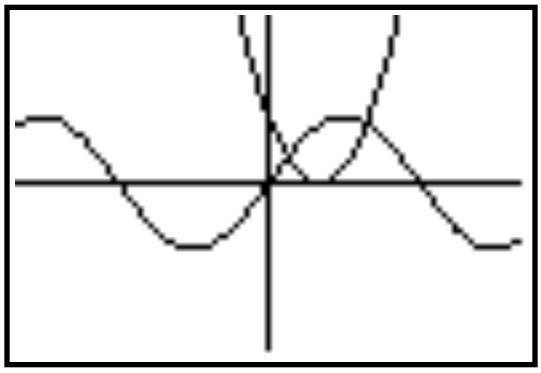
6. Solve the following system using a graphing calculator. Express all solutions as ordered pairs. **(3 marks)**

$$y = 4\sin\frac{\pi}{6}x$$

$$y = (x-2)^2$$

Sketch the graph in the viewing window below. State the functions that you entered to obtain your graph and your solution. Indicate the dimensions of the viewing window that will show enough of the graphs so that recognizable characteristics of each function and all intersection points are visible. (*Note*: Graph at least one period of the sine curve.)

Solution



$Y_1 = 4\sin\frac{\pi}{6}x$

$Y_2 = (x-2)^2$

← $\frac{1}{2}$ **mark** for graph of each curve (total **1 mark**)

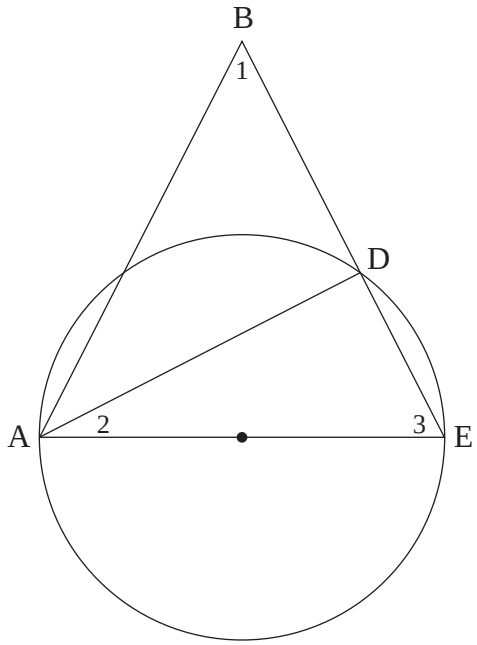
$x\ [-10,\ 10]$ $\quad$ $y\ [-10,\ 10]$

$(0.76,\ 1.54)$

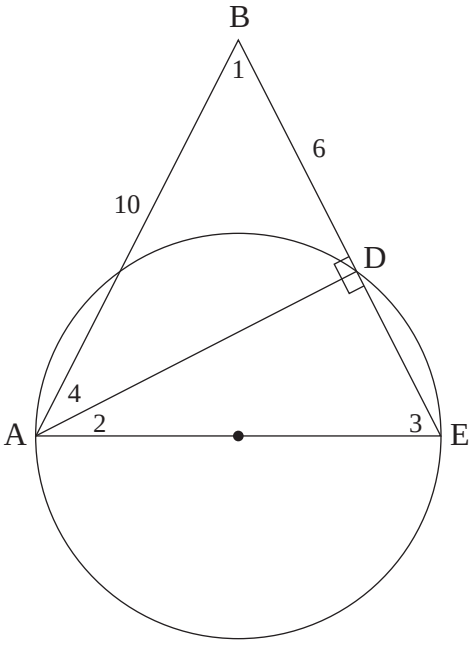
$(3.89,\ 3.57)$

} ← **2 marks**

7. In the diagram below, AE is a diameter, $AB = 10$, $BD = 6$, $\angle 1 = \angle 2$ and B, D, E are collinear. Find the measure of $\angle 3$. (Answer to the nearest degree.) **(3 marks)**



Solution



$\angle ADE = 90°$ ← **½ mark**

$\therefore \angle BDA = 90°$

and since $\angle 1 = \angle 2$

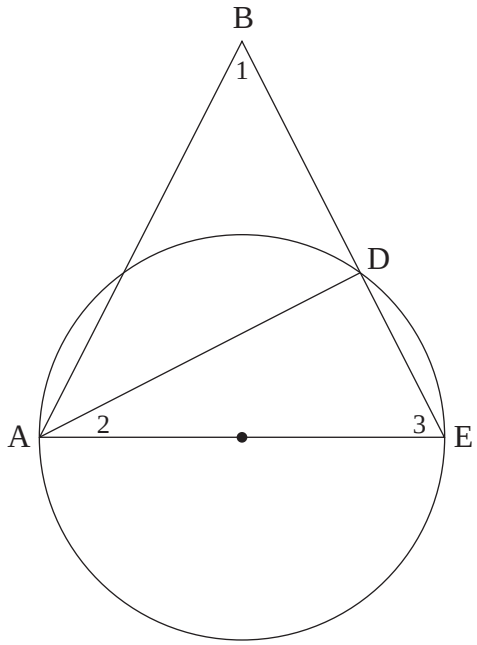
$\Rightarrow \angle 4 = \angle 3$ ← **½ mark**

$\sin \angle 4 = \frac{6}{10}$ ← **1 mark**

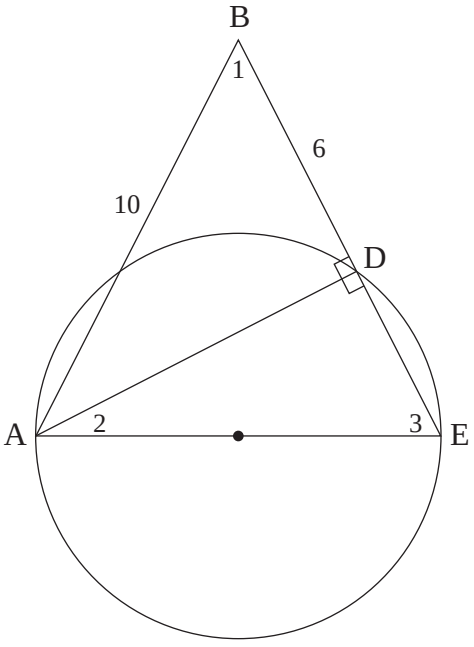
$\Rightarrow \angle 4 = 37°$ ← **½ mark**

$\therefore \angle 3 = 37°$ ← **½ mark**

7. In the diagram below, AE is a diameter, $AB = 10$, $BD = 6$, $\angle 1 = \angle 2$ and B, D, E are collinear. Find the measure of $\angle 3$. (Answer to the nearest degree.) **(3 marks)**



Alternate Solution



$\angle ADB = 90°$ ← **$\frac{1}{2}$ mark**

$\cos \angle 1 = \frac{6}{10}$ ← **1 mark**

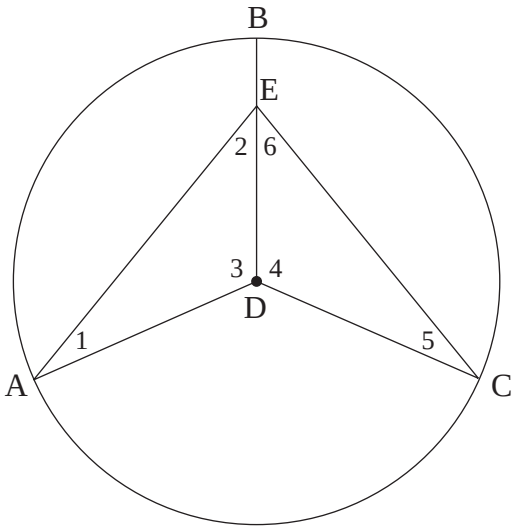
$\angle 1 = 53°$ ← **$\frac{1}{2}$ mark**

$\angle 3 = 37°$ ← **1 mark**

8. Complete the following proof. **(4 marks)**

Given: D is the centre of the circle
$\overset{\frown}{AB} = \overset{\frown}{BC}$
B, E, D are collinear

Prove: $\angle 1 = \angle 5$



Solution

Method 1:

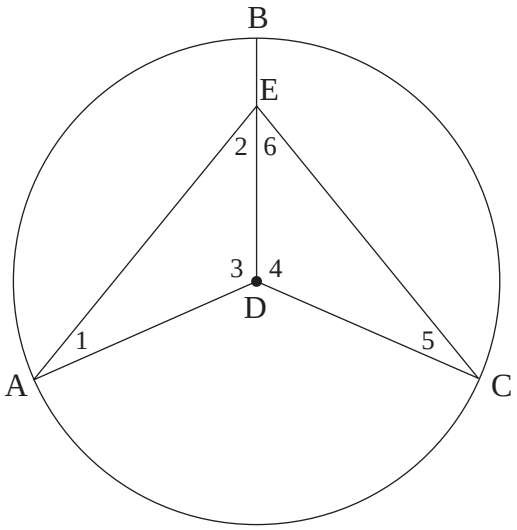
PROOF

Statement		Reason	
	$\overset{\frown}{AB} = \overset{\frown}{BC}$		given
$\frac{1}{2}$ **mark** $\rightarrow$	$\angle 3 = \angle 4$	**1 mark** $\rightarrow$	central $\angle$s on = arcs are =
	$DA = DC$	$\frac{1}{2}$ **mark** $\rightarrow$	radii =
	$DE = DE$	$\frac{1}{2}$ **mark** $\rightarrow$	same side
$\frac{1}{2}$ **mark** $\rightarrow$	$\Delta AED \cong \Delta CED$	$\frac{1}{2}$ **mark** $\rightarrow$	SAS
	$\angle 1 = \angle 5$	$\frac{1}{2}$ **mark** $\rightarrow$	CPCTC

8. Complete the following proof. **(4 marks)**

Given: D is the centre of the circle
$\overset{\frown}{AB} = \overset{\frown}{BC}$
B, E, D are collinear

Prove: $\angle 1 = \angle 5$



Alternate Solution 1

Method 1:

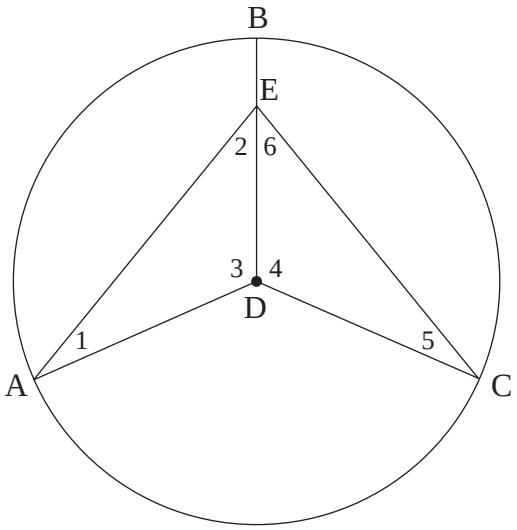
PROOF

Statement	Reason
$\overset{\frown}{AB} = \overset{\frown}{BC}$	given
join AB	
join CB	
AB = CB	$\frac{1}{2}$ **mark** $\rightarrow$ chords on = arcs are =
DB = DB	$\frac{1}{2}$ **mark** $\rightarrow$ same side
AD = CD	$\frac{1}{2}$ **mark** $\rightarrow$ radii are =
$\Delta ABD \cong \Delta CBD$	$\frac{1}{2}$ **mark** $\rightarrow$ SSS
$\angle 3 = \angle 4$	$\frac{1}{2}$ **mark** $\rightarrow$ CPCTC
ED = ED	$\frac{1}{2}$ **mark** $\rightarrow$ same side
$\frac{1}{2}$ **mark** $\rightarrow$ $\Delta ADE \cong \Delta CDE$	$\frac{1}{2}$ **mark** $\rightarrow$ SAS
$\angle 1 = \angle 5$	$\frac{1}{2}$ **mark** $\rightarrow$ CPCTC

8. Complete the following proof. **(4 marks)**

Given: D is the centre of the circle
$\overset{\frown}{AB} = \overset{\frown}{BC}$
B, E, D are collinear

Prove: $\angle 1 = \angle 5$



Alternate Solution 2

Method 1:

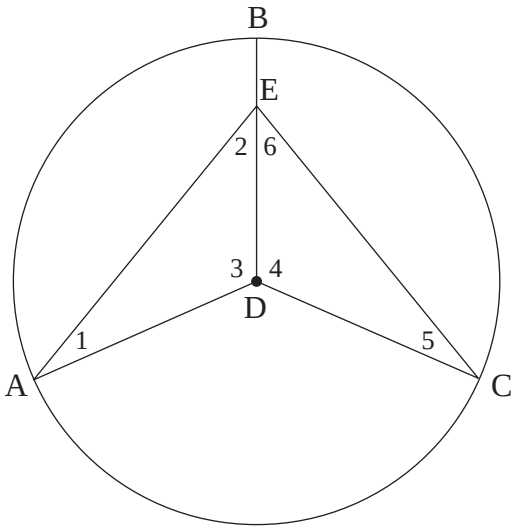
PROOF

	Statement		Reason
	$\overset{\frown}{AB} = \overset{\frown}{BC}$		given
	join AB		
	join CB		
	AB = CB	$\frac{1}{2}$ **mark** →	chords on = arcs are =
	DB = DB	$\frac{1}{2}$ **mark** →	same side
	AD = CD		radii are =
	$\Delta ABD \cong \Delta CBD$	$\frac{1}{2}$ **mark** →	SSS
	$\angle ABD = \angle CBD$		CPCTC
	BE = BE		same side
	$\Delta ABE \cong \Delta CBE$	$\frac{1}{2}$ **mark** →	SAS
	AE = CE		CPCTC
	DE = DE	$\frac{1}{2}$ **mark** →	same side
$\frac{1}{2}$ **mark** →	$\Delta ADE \cong \Delta CDE$	$\frac{1}{2}$ **mark** →	SSS
	$\angle 1 = \angle 5$	$\frac{1}{2}$ **mark** →	CPCTC

8. Complete the following proof. **(4 marks)**

Given: D is the centre of the circle
$\overset{\frown}{AB} = \overset{\frown}{BC}$
B, E, D are collinear

Prove: $\angle 1 = \angle 5$



Solution

Method 2:

either $\frac{1}{2}$ **mark**

Since $\overset{\frown}{AB} = \overset{\frown}{BC}$, $\angle 3 = \angle 4$ since central $\angle$s on = arcs are =

$\frac{1}{2}$ **mark** $\frac{1}{2}$ **mark**

DA = DC since radii are = $\leftarrow \frac{1}{2}$ **mark**

and DE = DE $\Rightarrow$ $\Delta AED \cong \Delta CED$ by SAS

$\frac{1}{2}$ **mark** $\frac{1}{2}$ **mark** $\frac{1}{2}$ **mark**

$\therefore \angle 1 = \angle 5$ by CPCTC $\leftarrow \frac{1}{2}$ **mark**

END OF KEY