Principles of Mathematics 12

June 2000 Provincial Examination

ANSWER KEY / SCORING GUIDE

CURRICULUM:

Organizers	Sub-Organizers
1. Problem Solving	A Problem Set
2. Patterns and Relations	B Sequences and Series
	C Polynomials
	D Logarithms and Exponents
	E Quadratic Relations
	F Quadratic Systems
3. Shape and Space	G Trigonometry
	H Geometry

Part A: Multiple Choice

Q	K	C	CO	PLO
1.	D	U	2	C5
2.	D	K	2	C1
3.	C	U	2, 1	C6, A7
4.	B	H	2	C9, C1
5.	D	H	2, 1	C1, A7
6.	D	K	2	E5
7.	B	U	2	E2
8.	A	U	2	F4
9.	B	U	2	F1
10.	A	U	2	F1
11.	A	U	2	E6
12.	B	H	2	E4
13.	D	H	2	E5
14.	A	K	2	D4
15.	C	U	2	D5
16.	A	U	2	D2
17.	C	U	2	D5
18.	A	U	2	D5
19.	B	H	2	D1
20.	D	H	2	D5
21.	A	K	2	B1
22.	C	U	2	B2
23.	B	U	2	B5
24.	C	U	2	B4
25.	D	U	2	B2
26.	B	U	2	B4
27.	D	H	2	B4
28.	C	H	2	B6, D5
29.	A	K	3	G5
30.	C	K	3	G5
31.	B	U	3	G5
32.	B	U	3	G1
33.	C	U	3	G3
34.	D	U	3	G3
35.	B	U	3, 1	G3, A7
36.	C	H	3	G5
37.	B	H	3	G10
38.	A	H	3	G9
39.	D	U	3	H2
40.	C	U	3	H1, H2
41.	D	U	3	H1, H2
42.	D	H	3	H4
43.	B	U	1	A3
44.	C	U	1	A3
45.	B	H	1	A3

Multiple Choice = 45 marks

Part B: Written Response

Q	B	C	S	CO	PLO
1.	1	U	3	2	F3
2.	2	U	3	2, 1	C4, C6, A7
3.	3	U	3	2, 1	D2, A7
4.	4	U	3	2	E7
5.	5	U	3	3	G8
6.	6	U	3	3	H4
7.	7	H	3	1	A3
8.	8	U	4	3	H2

Written Response = 25 marks

Multiple Choice = 45 (45 questions)
Written Response = 25 (8 questions)
EXAMINATION TOTAL = 70 marks

LEGEND:

Q = Question Number
K = Keyed Response
C = Cognitive Level
B = Score Box Number
S = Score
CO = Curriculum Organizer
PLO = Prescribed Learning Outcome

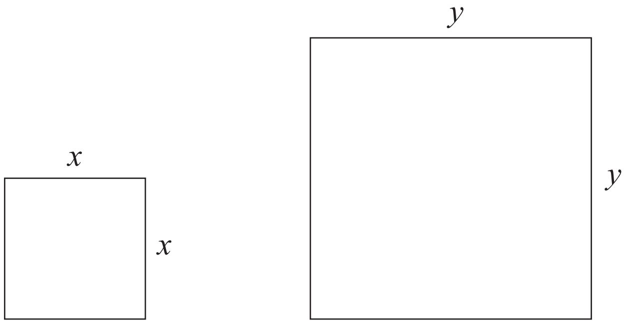
PART B: WRITTEN RESPONSE

Value: 25 marks **Suggested Time: 45 minutes**

INSTRUCTIONS:	Rough-work space has been incorporated into the space allowed for answering each question. You may not need all the space provided to answer each question. Where required, place the final answer for each question in the space provided. If, in a justification, you refer to information produced by the calculator, this information must be presented clearly in the response. For example, if a graph is used in the solution of the problem, it is important to sketch the graph, showing its general shape and indicating the appropriate window dimensions. When using the calculator, you should provide a decimal answer that is correct to **at least two decimal places** (unless otherwise indicated). Such rounding should occur **only** in the final step of the solution. **Full marks will NOT be given for the final answer only.**

1. Solve the following problem algebraically. **(3 marks)**

The sum of the areas of two separate squares is 234 cm^2. The sum of their perimeters is 72 cm. Determine the measure of the sides of each square.



Solution

$$x^2 + y^2 = 234 \qquad \leftarrow \tfrac{1}{2} \textbf{ mark}$$

$$4x + 4y = 72 \qquad \leftarrow \tfrac{1}{2} \textbf{ mark}$$

$$\therefore \; x + y = 18$$

$$y = 18 - x$$

$$x^2 + (18 - x)^2 = 234 \qquad \leftarrow \tfrac{1}{2} \textbf{ mark}$$

$$x^2 + 324 - 36x + x^2 = 234$$

$$\left.\begin{aligned} 2x^2 - 36x + 90 &= 0 \\ x^2 - 18x + 45 &= 0 \end{aligned}\right\} \leftarrow \tfrac{1}{2} \textbf{ mark}$$

$$\left.\begin{array}{ll} (x-15)(x-3) = 0 & \\ x = 15 & x = 3 \\ \downarrow & \\ y = 3 & y = 15 \end{array}\right\} \text{indication of method other than factoring eg., graphing calc.}$$

$\therefore$ the squares have sides of lengths 15 cm and 3 cm

$$\underset{\tfrac{1}{2}\textbf{ mark}}{\uparrow} \qquad \underset{\tfrac{1}{2}\textbf{ mark}}{\uparrow}$$

2. When $2x^3 - 8x^2 + kx + 18$ is divided by $x + 2$, the remainder is –14. Find k, then use a graphing calculator to find all real roots of $2x^3 - 8x^2 + kx + 18 = 0$. (**Note**: It is not necessary to show the viewing window.) **(3 marks)**

Solution

$\left(2x^3 - 8x^2 + kx + 18\right) \div (x + 2)$ has remainder -14

$$\overbrace{\overset{\downarrow}{2(-2)^3} - \overset{\downarrow}{8(-2)^2} + \overset{\downarrow}{k(-2)}}^{\frac{1}{2}\textbf{ mark}} + 18 = \overset{\frac{1}{2}\textbf{ mark}}{\overset{\downarrow}{-14}}$$

$$-16 + -32 - 2k + 18 = -14$$

$$-2k = 16$$

$$k = -8 \quad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

OR

$$\begin{array}{r|rrrr} \overset{\frac{1}{2}\textbf{ mark}}{\overset{\downarrow}{-2}} & 2 & -8 & k & 18 \\ & & -4 & 24 & -2k - 48 \\ \hline & 2 & -12 & k + 24 & -2k - 30 \end{array}$$

$$-2k - 30 = -14 \quad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

$$-2k = 16$$

$$k = -8 \quad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

$$2x^3 - 8x^2 - 8x + 18 = 0$$

$$x = \underset{\frac{1}{2}\textbf{ mark}}{\underset{\uparrow}{-1.66}}, \quad \underset{\frac{1}{2}\textbf{ mark}}{\underset{\uparrow}{1.22}}, \quad \underset{\frac{1}{2}\textbf{ mark}}{\underset{\uparrow}{4.44}}$$

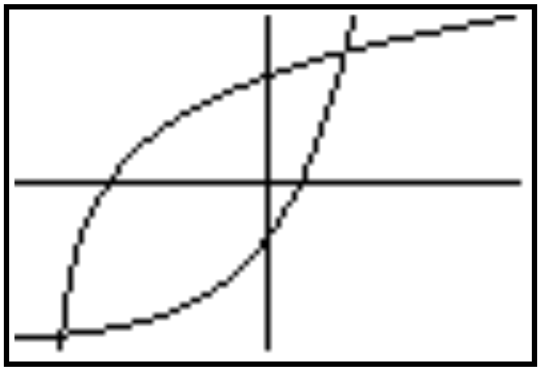
3. Solve the following system using a graphing calculator. **(3 marks)**

$$y = \log_2(x+4)$$

$$y = 2^{x+1} - 3$$

Sketch the graph in the viewing window below. State the function(s) that you entered to obtain your graph and your solution. Indicate the dimensions of the viewing window that will show enough of the graph so that recognizable characteristics of the function(s) and all intersection points are visible.

Solution



$$\left.\begin{array}{l} Y_1 = \dfrac{\log(x+4)}{\log(2)} \\ Y_2 = 2^{x+1} - 3 \end{array}\right\} \leftarrow \tfrac{1}{2} \textbf{ mark} \text{ for equations}$$

← $\frac{1}{2}$ **mark** for graph

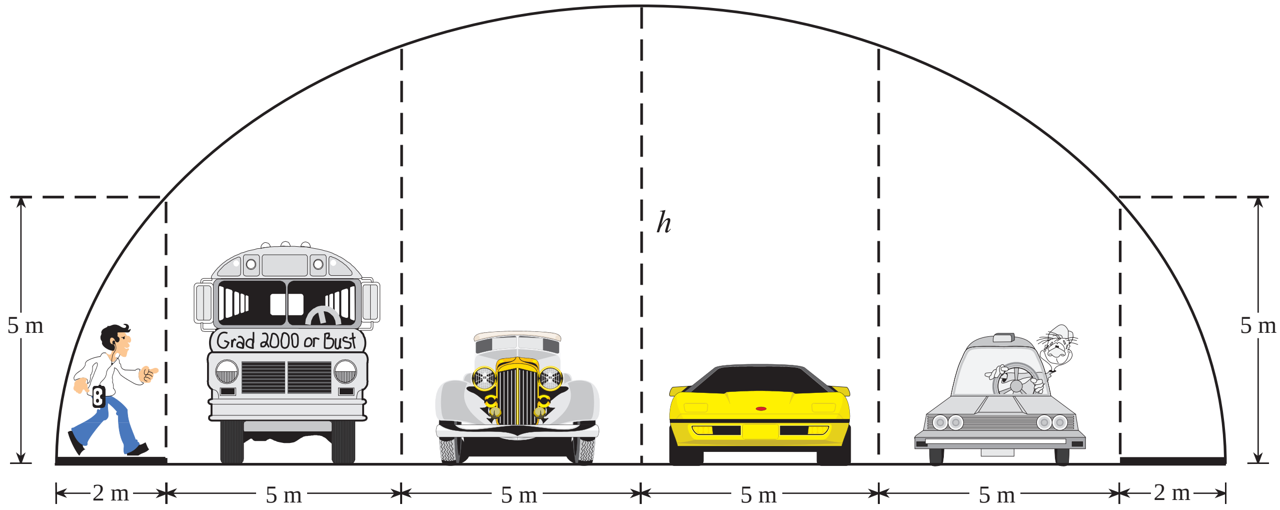
alternatively: one graph plus its equation = $\frac{1}{2}$ **mark**

$x \; [-4.7, \; 4.7]$ $\quad y \; [-3.1, \; 3.1]$ ← $\frac{1}{2}$ **mark** for window dimensions

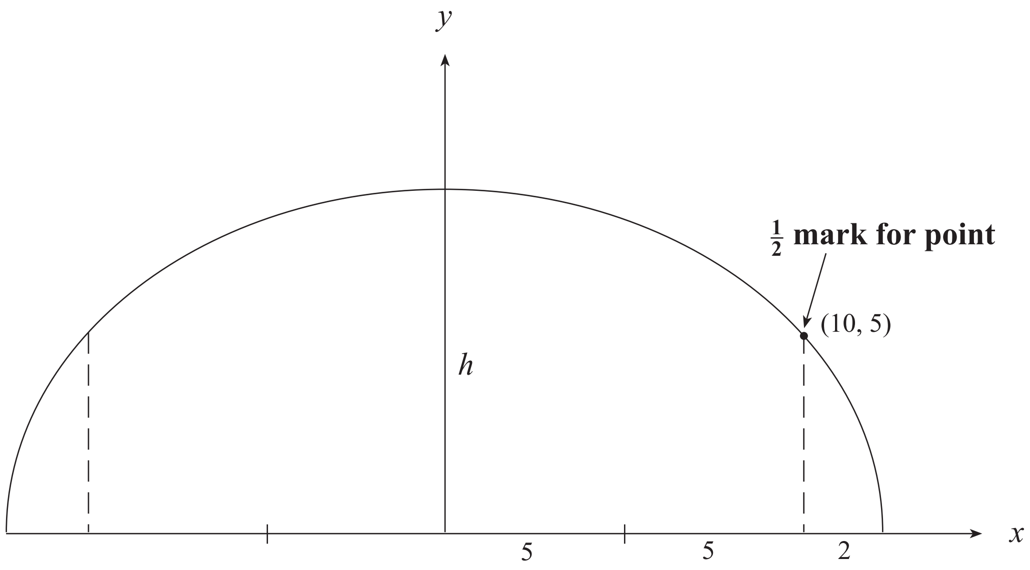
Therefore the solution to the system is:

$(1.44, \; 2.44)$ $\quad (-3.86, \; -2.86)$ ← **1$\frac{1}{2}$ marks** (deduct $\frac{1}{2}$ **mark** if one solution only; deduct **1 mark** if only *x*-coordinates are given)

4. A tunnel is semi-elliptical in shape. It has four traffic lanes, each 5 m wide, as well as two service walkways, each 2 m wide, as shown in the diagram. The tunnel has a height of 5 m at the edge of the roadway. Determine the height, h, of the tunnel at its highest point. **(3 marks)**



Solution



½ mark for point

½ mark for ellipse equation

½ mark → $$\frac{x^2}{12^2} + \frac{y^2}{h^2} = 1$$

½ mark
↓

$$\frac{10^2}{12^2} + \frac{5^2}{h^2} = 1$$

$$\left.\begin{aligned} \frac{25}{h^2} &= 1 - \frac{100}{144} \\ \frac{25}{h^2} &= \frac{11}{36} \\ h^2 &= \frac{(36)(25)}{11} \\ &= 81.81 \end{aligned}\right\}$$

any one of these
for **½ mark**

$h = 9.05$ m ← **½ mark**

5. Prove the identity: **(3 marks)**

$$\frac{1}{\sec\theta+\tan\theta} \qquad = \qquad \frac{1-\sin\theta}{\cos\theta}$$

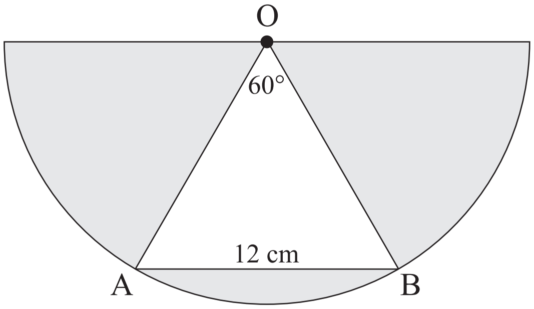
Solution

$$\frac{1}{\sec\theta+\tan\theta} \qquad = \qquad \frac{1-\sin\theta}{\cos\theta}$$

LEFT SIDE	RIGHT SIDE
$= \left.\frac{\left(\quad 1 \quad\right)}{\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)}\frac{\cos\theta}{\cos\theta}\right\} \leftarrow \tfrac{1}{2}$ **mark** $\leftarrow \tfrac{1}{2}$ **mark**	
$\tfrac{1}{2}$ **mark** $\rightarrow = \frac{\cos\theta}{(1+\sin\theta)}\,\frac{(1-\sin\theta)}{(1-\sin\theta)} \leftarrow \tfrac{1}{2}$ **mark**	
$= \frac{\cos(1-\sin\theta)}{1-\sin^2\theta} \leftarrow \tfrac{1}{2}$ **mark**	
$= \frac{\cos(1-\sin\theta)}{\cos^2\theta} \leftarrow \tfrac{1}{2}$ **mark**	
$= \frac{1-\sin\theta}{\cos\theta}$	

LS = RS

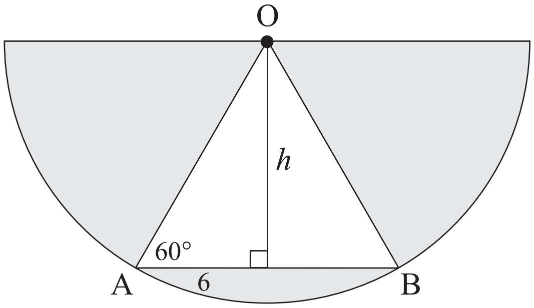
6. In a semi-circle with centre O, $\angle AOB = 60°$ and $AB = 12$ cm. Determine the area of the shaded region. **(3 marks)**



Solution

1 mark →

$$\begin{cases} \text{since } AO = OB,\ \angle A = \angle B \\ \text{so } \angle A = \angle B = 60°,\ \text{thus } AO = 12 \\ \frac{h}{6} = \tan 60° \\ h = 10.3923 \end{cases}$$



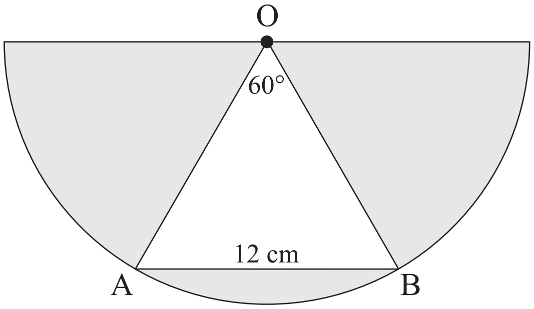
½ mark → $\therefore\ \text{area}_{\Delta AOB} = \frac{1}{2}(12)10.3923 = 62.3538$

1 mark → $\therefore\ \text{area}_{\text{shaded}} = \frac{\pi(12)^2}{2} - 62.3538 = 163.84$

½ mark → $\therefore$ area of shaded region is $163.84\,\text{cm}^2$

$\left(\text{Exact answer: } 72\pi - 36\sqrt{3}\ \text{cm}^2\right)$

6. In a semi-circle with centre O, $\angle AOB = 60°$ and $AB = 12$ cm. Determine the area of the shaded region. **(3 marks)**



Alternate Solution

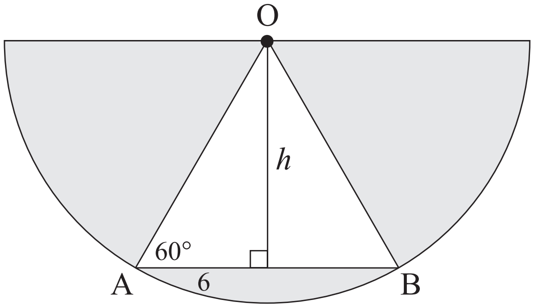
1 mark →

$$\text{since } AO = OB,\ \angle A = \angle B$$

$$\text{so } \angle A = \angle B = 60°,\ \text{thus } AO = 12$$

$$\therefore\ 6^2 + h^2 = 12^2$$

$$h = 10.3923,\ 6\sqrt{3},\ \sqrt{108}$$



$\frac{1}{2}$ mark → $\therefore\ \text{area}_{\Delta AOB} = \frac{1}{2}(12)10.3923 = 62.3538$

1 mark → $\therefore\ \text{area}_{\text{shaded}} = \frac{\pi(12)^2}{2} - 62.3538 = 163.84$

$\frac{1}{2}$ mark → $\therefore$ area of shaded region is $163.84\,\text{cm}^2$

$\left(\text{Exact answer: } 72\pi - 36\sqrt{3}\ \text{cm}^2\right)$

7. Solve for y and state all restrictions on x and y. **(3 marks)**

$$\frac{1}{\log_y 3} = \log_{\frac{1}{3}} 27 + 2\log_3 x$$

Solution

$$\frac{1}{\log_y 3} = \log_{\frac{1}{3}} 27 + 2\log_3 x$$

$$\overset{\frac{1}{2}\textbf{ mark}\ \downarrow}{\boxed{\frac{1}{\frac{\log 3}{\log y}} = \frac{\log y}{\log 3} = \log_3 y}} = \overset{\frac{1}{2}\textbf{ mark}\ \downarrow}{\boxed{-\log_3 27}} + \log_3 x^2$$

also →

$$\boxed{\begin{array}{l} \log_3 \frac{1}{27} = \log_3 3^{-3} = \log_3 27^{-1} = -\log_3 27 \\ = -\log_3 3^3 = -3 = \frac{\log 27}{-\log 3} = -3\log_3 3 \end{array}}$$

$$\left.\begin{array}{ll} \log_3 y = \log_3\left(\frac{x^2}{27}\right) & \leftarrow \frac{1}{2}\textbf{ mark} \\ y = \frac{x^2}{27} & \leftarrow \frac{1}{2}\textbf{ mark} \end{array}\right\} \leftarrow \frac{1}{2}\textbf{ mark}\ \text{for } \textbf{\textit{concept}} \text{ of equating arguments even if arguments not correct}$$

$$y > 0\,, \quad y \neq 1 \quad \} \leftarrow \frac{1}{2}\textbf{ mark}$$

$$x > 0 \quad \} \leftarrow \frac{1}{2}\textbf{ mark}$$

(**Note**: It is not necessary for students to state $x \neq \sqrt{27}$)

Students should choose one or the other method of proof.

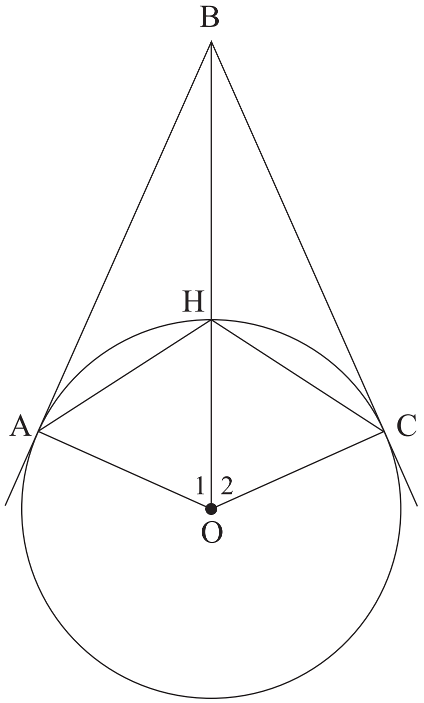
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Solution

Paragraph proof method:

Since O is the centre, AO = CO because radii = .

Since AB and CB are tangents, BA = BC since tangents from an external point are = .

With common side BO then ΔBAO ≅ ΔBCO by SSS. So ∠1 = ∠2.

Therefore, AH = CH since they are chords on = central ∠s.

Students should choose one or the other method of proof.

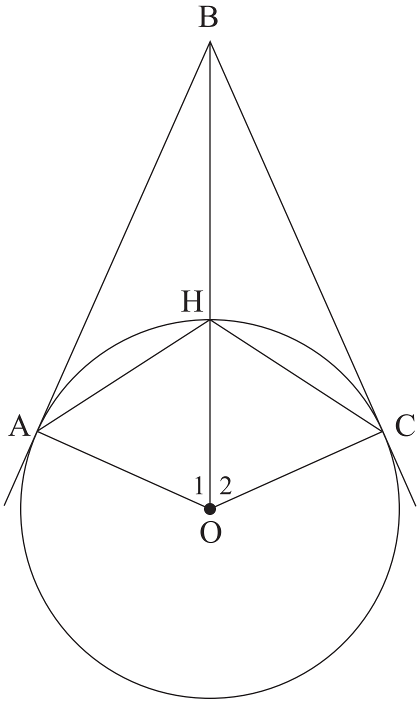
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 1

Paragraph proof method:

Since AB and CB are tangents, BA = BC because tangents from an external point are =,

and $\angle BAO = \angle BCO = 90°$

because tangent ⊥ to radius, and OA = OC since radii are = .

Therefore, $\Delta BAO \cong \Delta BCO$ by SAS, so $\angle 1 = \angle 2$.

Thus, AH = CH since they are chords on = central ∠s.

Students should choose one or the other method of proof.

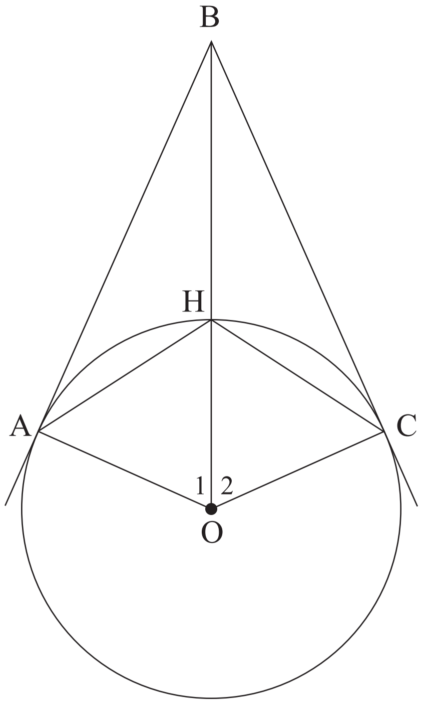
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 2

Paragraph proof method:

Since AB and CB are tangents, BA = BC because tangents from an external point are equal. AO = CO since radii are equal and ∠BAO = ∠BCO = 90° since tangents are perpendicular to radii. Therefore ΔBAO ≅ ΔBCO by SAS.

Since ∠1 = ∠2 and OH is the common side then ΔAHO ≅ ΔCHO by SAS.

So AH = CH.

Students should choose one or the other method of proof.

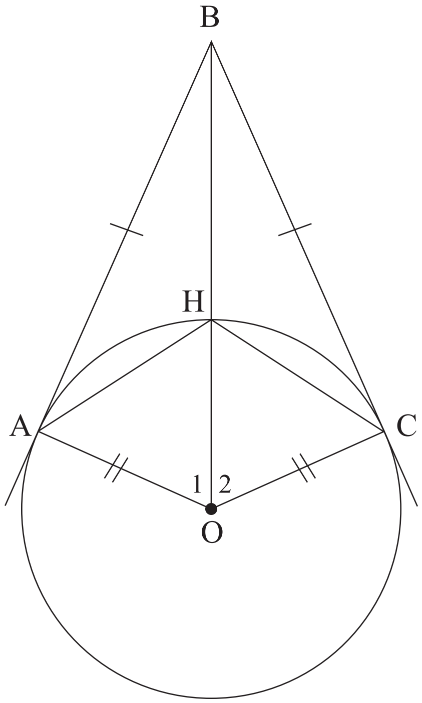
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Solution

Two-column proof method:

	STATEMENT	REASON
2 $\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	BO = BO	same side
	ΔBAO ≅ ΔBCO	SSS
1 $\frac{1}{2}$ marks	∠1 = ∠2	CPCTC
	AH = CH	chords opposite = central ∠s are =

Students should choose one or the other method of proof.

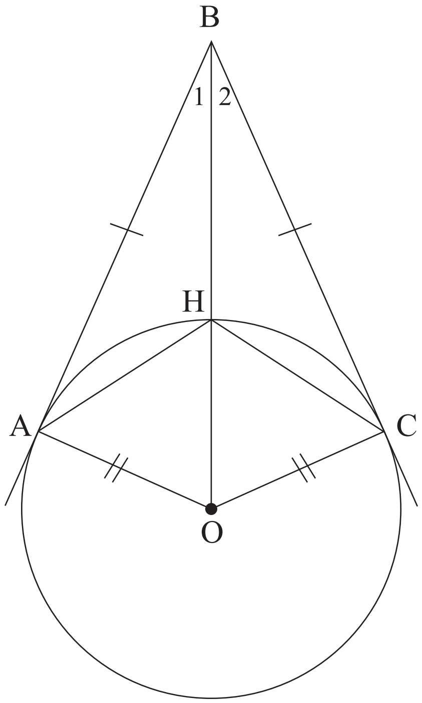
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 1

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	BO = BO	same side
	ΔBAO ≅ ΔBCO	SSS
$1\frac{1}{2}$ marks	∠1 = ∠2	CPCTC
	BH = BH	same side
	ΔABH ≅ ΔCBH	SAS
	AH = CH	CPCTC

Students should choose one or the other method of proof.

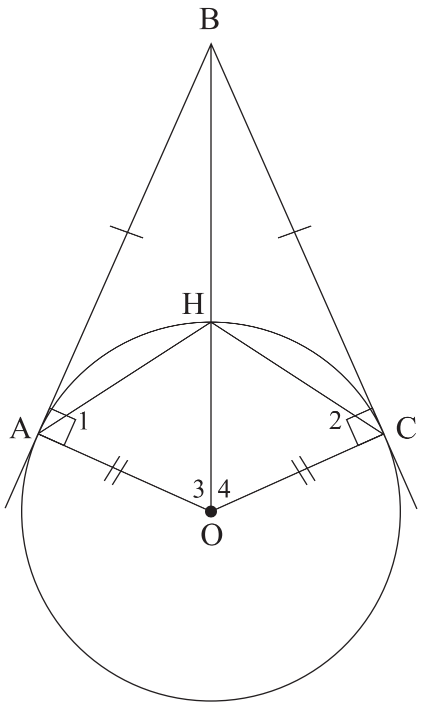
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 2

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	∠1 = 90°	tangent ⊥ radius
	∠2 = 90°	tangent ⊥ radius
	∠1 = ∠2	both = 90°
	ΔBAO ≅ ΔBCO	SAS
$1\frac{1}{2}$ marks	∠3 = ∠4	CPCTC
	AH = CH	chords opposite = central ∠s are =

Students should choose one or the other method of proof.

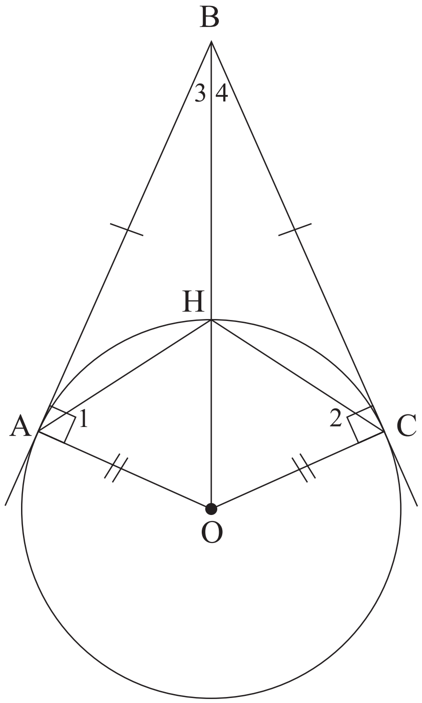
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 3

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	$\angle 1 = 90°$	tangent ⊥ radius
	$\angle 2 = 90°$	tangent ⊥ radius
	$\angle 1 = \angle 2$	both = 90°
	$\Delta BAO \cong \Delta BCO$	SAS
$1\frac{1}{2}$ marks	$\angle 3 = \angle 4$	CPCTC
	BH = BH	same side
	$\Delta AHB \cong \Delta CHB$	SAS
	AH = CH	CPCTC

Students should choose one or the other method of proof.

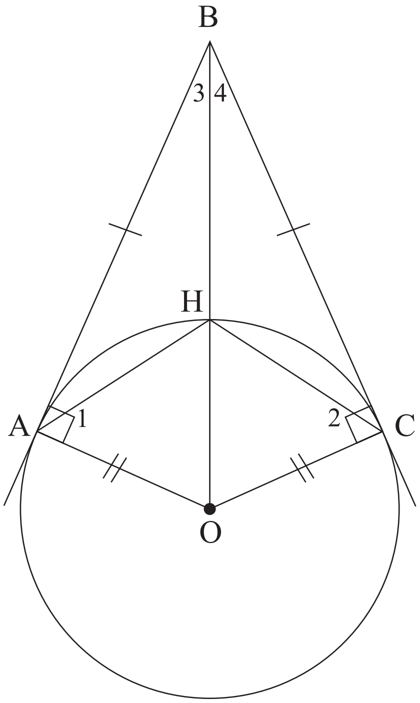
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 4

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	∠1 = 90°	tangent ⊥ radius
	∠2 = 90°	tangent ⊥ radius
	∠1 = ∠2	both = 90°
	BO = BO	same side
	ΔABO ≅ ΔCBO	HL
$1\frac{1}{2}$ marks	∠3 = ∠4	CPCTC
	BH = BH	same side
	ΔABH ≅ ΔCBH	SAS
	AH = CH	CPCTC

Students should choose one or the other method of proof.

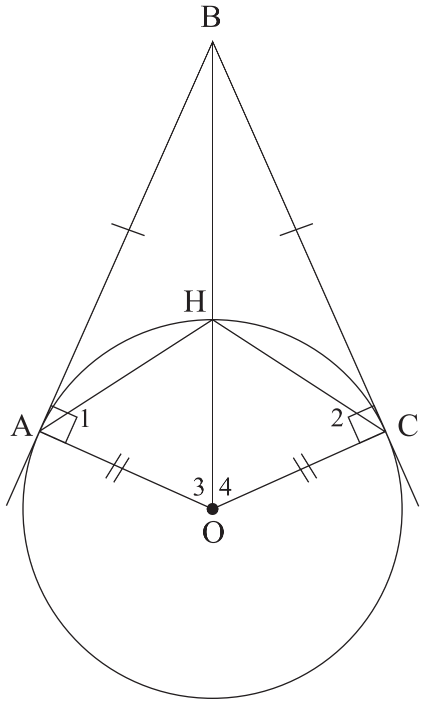
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 5

Two-column proof method:

	STATEMENT	REASON
2 $\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	∠1 = 90°	tangent ⊥ radius
	∠2 = 90°	tangent ⊥ radius
	∠1 = ∠2	both = 90°
	BO = BO	same side
	ΔABO ≅ ΔCBO	HL
1 $\frac{1}{2}$ marks	∠3 = ∠4	CPCTC
	AH = CH	chords opposite = central ∠s are =

Students should choose one or the other method of proof.

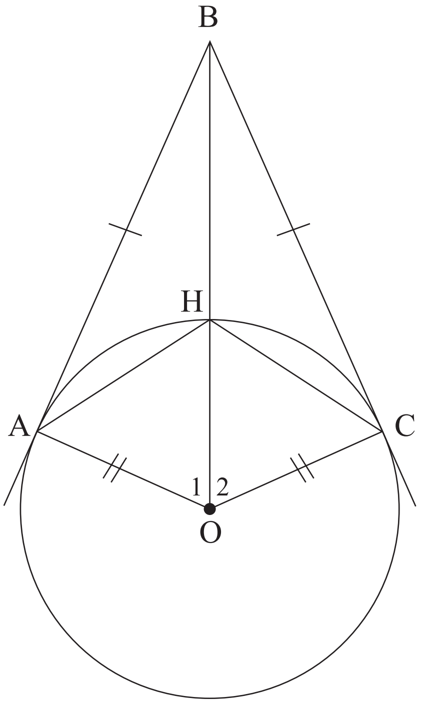
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 6

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	BO = BO	same side
	ΔBAO ≅ ΔBCO	SSS
$1\frac{1}{2}$ marks	$\angle 1 = \angle 2$	CPCTC
	OH = OH	same side
	ΔAOH ≅ ΔCOH	SAS
	AH = CH	CPCTC

Students should choose one or the other method of proof.

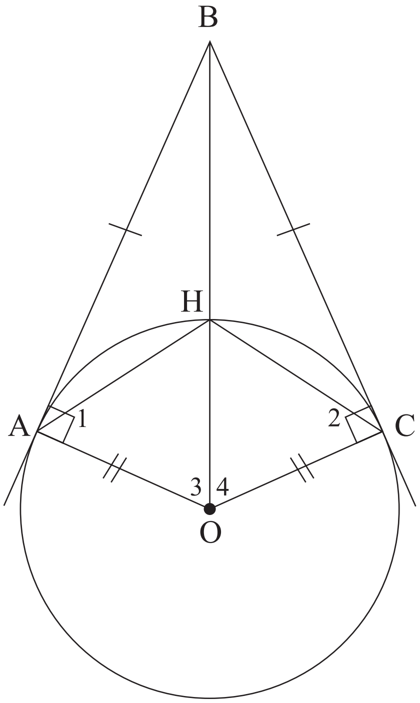
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 7

Two-column proof method:

	STATEMENT	REASON
$2\frac{1}{2}$ marks	AB and CB are tangents	given
	AB = CB	tangents from external point are =
	join OC	
	join OA	
	OA = OC	radii are =
	∠1 = 90°	tangent ⊥ radius
	∠2 = 90°	tangent ⊥ radius
	∠1 = ∠2	both = 90°
	ΔBAO ≅ ΔBCO	SAS
$1\frac{1}{2}$ marks	∠3 = ∠4	CPCTC
	OH = OH	same side
	ΔAOH ≅ ΔCOH	SAS
	AH = CH	CPCTC

Students should choose one or the other method of proof.

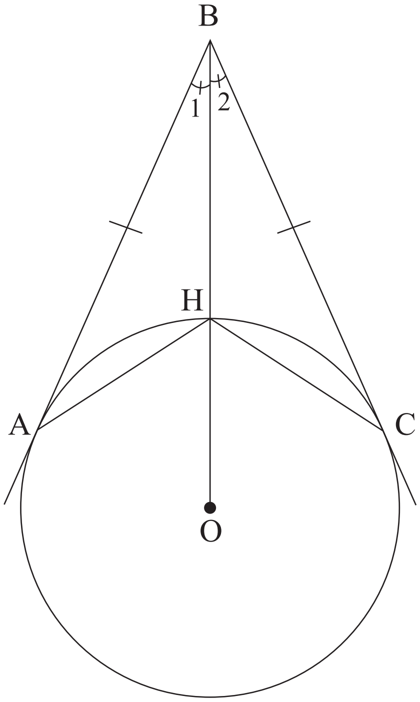
8. Complete the proof. **(4 marks)**

Diagram clarification: O is the centre
B, H, O are collinear

Given: AB and CB are tangents

Prove: AH = CH

Note: Students are encouraged to number angles.



Alternate Solution 8

Two-column proof method:

STATEMENT	REASON
AB and CB are tangents	given
AB = CB	tangents from external point are =
∠1 = ∠2	line from the centre to the external point will bisect the angle formed by the tangents at the external point
BH = BH	same side
ΔABH ≅ ΔCBH	SAS
AH = CH	CPCTC

END OF KEY